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A *hp*-discontinuous Galerkin method for the time-dependent Maxwell's equation: a priori error estimate

C. Daveau · A. Zaghdani

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Abstract A discontinuous Galerkin method for the numerical approximation for the time-dependent Maxwell's equations in "stable medium" with supraconductive boundary, is introduced and analysed. its hp-analysis is carried out and error estimates that are optimal in the meshsize h and slightly suboptimal in the approximation degree p are obtained.

Keywords Discontinuous Galerkin method · Wave equation · A priori error estimate

Mathematics Subject Classification (2000) 65N30

1 Introduction

Electromagnetism is one application of the discontinuous Galerkin method, among many other areas. In [1], the discontinuous Galerkin method with solutions that are exactly divergence-free inside each element, is developped for numerically solving the Maxwell equations. In [2], M. Grote, A. Schneebeli and D. Schötzau propose and analyse the symmetric interior penalty discontinuous Galerkin method for the spatial discretization of Maxwell's equations in second order form.

Here, we consider a nonsymmetric interior penalty discontinuous Galerkin method to approximate in space an initial boundary value problem derived from Maxwell's equations in vacuum with perfect electric conductor boundary

$$\frac{\partial^2 u}{\partial t^2} + c^2 \nabla \times (\nabla \times u) = f, \qquad \nabla \cdot u = 0 \quad \text{in } \Omega \times I; \tag{1.1}$$

C. Daveau (✉) · A. Zaghdani
CNRS, UMR 8088, Département de Mathématiques, Université de Cergy-Pontoise, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France
e-mail: christian.daveau@math.u-cergy.fr

$$
\begin{aligned}
&n \times u(x,t) = 0 \quad \text{on } \partial\Omega \times I, \\
&u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{on } \Omega.
\end{aligned} \tag{1.2}
$$

Here Ω is a convex polyhedron included in $\mathbb{R}^3$, $I = [0, t^*] \subset \mathbb{R}$, u_0 and u_1 are in $H_0(\nabla\times, \Omega) \cap H(\nabla \cdot 0, \Omega)$ and f is defined on $\Omega \times I$. Physically, u is the electric field, and f is related to a current density. Moreover $\mu_0\varepsilon_0 c^2 = 1$ where $\mu_0 \approx 4\pi 10^{-7}\ \mathrm{H \cdot m^{-1}}$ and $\varepsilon_0 \approx (36\pi 10^9)^{-1}\mathrm{F \cdot m^{-1}}$ are respectively the magnetic permeability and the electric permittivity in vacuum. If we assume that the domain Ω is "stable medium" with boundary and if u is the exact solution of (1.1) and (1.2) then u and $\nabla \times u$ belong to $H^1(\Omega)^3$. For the notations, if I is an interval, X is a function space and ϕ is a function on $\Omega \times I$ then $\|\phi\|_{L^p(I,X)}$ denotes the norm in $L^p(I)$ of the function $t \to \|\phi(\cdot,t)\|_X$. $L^p(X)$ is short for $L^p(I, X)$.

Let Π_h be a partition of Ω into tetrahedra and consider the same spaces and notations as in [4].

Faces We define and characterise the faces of the triangulation Π_h. An interior face of Π_h is defined as the (non-empty) two-dimensional interior of $\partial K_1 \cap \partial K_2$, where K_1 and K_2 are two adjacents elements of Π_h. A boundary face of Π_h is defined as the (non-empty) two-dimensional interior of $\partial K \cap \partial\Omega$, where K is a boundary element of Π_h. We denote by F_h^I the union of all interior faces of Π_h, by F_h^D the union of all boundary faces of Π_h and by F_h the union of all faces of Π_h. Furthermore we identify F_h^D to $\partial\Omega$ since Ω is a polyhedron.

Traces Let $H^s(\Pi_h) = \{v : v_{|K} \in H^s(K)\ \forall K \in \Pi_h\}$ for $s > \frac{1}{2}$ endowed with the norm $\|v\|_{s,\Pi_h}^2 = \sum_{K\in\Pi_h} \|v\|_{s,K}^2$. Then the elementwise traces of functions in $H^s(\Pi_h)$ belongs to $\mathrm{TR}(F_h) = \Pi_{K\in\Pi_h} L^2(\partial K)$; they are double-valued on F_h^I and single-valued on F_h^D. The space $L^2(F_h)$ can be identified with the functions in $\mathrm{TR}(F_h)$ for which the two traces values coincide.

Traces operators Let us introduce the following traces operators for piecewise smooth functions. First, let $w \in \mathrm{TR}(F_h)^3$ and $e \subset F_h$. If e is an interior face in F_h^I, we denote by K_1 and K_2 the elements sharing e, by n_i the normal unit vector pointing exterior to K_i and we set $\omega_i = \omega_{|\partial K_i}$, $i = 1, 2$. We define the *average*, and the *tangential* and *normal jumps* of w at $x \in e$ as

$$
\{\omega\} = \frac{\omega_1 + \omega_2}{2}, \quad [\omega]_T = n_1 \times \omega_1 + n_2 \times \omega_2 \quad \text{and} \quad [\omega]_N = n_1 \cdot \omega_1 + n_2 \cdot \omega_2.
$$

If $e \subset F_h^D$, we set for $x \in e$

$$
\{\omega\} = \omega, \quad [\omega]_T = n \times \omega \quad \text{and} \quad [\omega]_N = n \cdot \omega.
$$

If $w \in H(\nabla\times, \Omega)$, then, for all $e \subset F_h^I$ the jump condition $[w]_T = 0$ holds true in $L^2(e)^3$ and if $w \in H(\nabla\cdot, \Omega)$, then, for all $e \subset F_h^I$ the jump condition $[w]_N = 0$ holds true in $L^2(e)$.

For $e \subset F_h$, we denote by $\langle \cdot, \cdot \rangle_e$ the scalar product in $L^2(e)^3$ or $L^2(e)$, furthermore if F_h^D is identified to $\partial\Omega$, we identify $\sum_{e \subset F_h^D} \langle \cdot, \cdot \rangle_e$ to $\langle \cdot, \cdot \rangle$, the scalar product in $L^2(\partial\Omega)^3$ or $L^2(\partial\Omega)$.

Finite element spaces In order to define the average of $\nabla \times u$, we set for $s > \frac{1}{2}$, $H^s(\nabla\times, \Pi_h) := \{v : v_{|K} \in H^s(K)^3 \text{ and } \nabla \times (v_{|K}) \in H^s(K)^3, \ \forall K \in \Pi_h\}$. Let $p = (p_K)_{K \in \Pi_h}$ be a degree vector that assigns to each element $K \in \Pi_h$ a polynomial approximation order $p_K \geq 1$. The generic *hp*-finite element space of piecewise polynomials is given by

$$S^p(\Pi_h) = \{u \in L^2(\Omega) : u_{|K} \in S^{p_K}(K), \ \forall K \in \Pi_h\}$$

where $S^{p_K}(K)$ is the space of real polynomials of degree at most p_K in K.

We also set $\Sigma_h := S^p(\Pi_h)^3$.

Now, fix a face $e \subset F_h$ and define the local parameters h, p by $h := \min(h_K, h_{K'})$, $p = \max(p_K, p_{K'})$ in the case of interior faces and $h := h_K$, $p = p_K$ in the case of boundary faces [3].

2 Discretization and a priori error estimate

In order to derive a weak formulation of (1.1)–(1.2), we note that formulas (1) in [4] implies that for any u with $\nabla \times u \in H(\nabla\times, \Omega)$

$$c^2(\nabla \times (\nabla \times u), v) = c^2(\nabla \times u, \nabla \times v) + a(u, v)$$

where

$$a(u, v) = c^2\langle n \times (\nabla \times u), v\rangle - c^2 \sum_{e \subset F_h^I} \langle [v]_T, \{\nabla \times u\}\rangle_e.$$

Now, we introduce the penalty term via the form

$$J^\sigma(u, v) = \sum_{e \subset F_h^I} \langle \sigma[u]_N, [v]_N\rangle_e + \sum_{e \subset F_h} \langle \sigma[u]_T, [v]_T\rangle_e, \quad u, v \in H^1(\nabla\times, \Pi_h)$$

where $\sigma := \kappa p^2/h$ is a stabilization parameter and κ is a constant supposed ≥ 1. We also define

$$A(u, v) = c^2(\nabla \times u, \nabla \times v) + a(v, u) - a(u, v) + J(u, v),$$
$$B(u, v) = A(u, v) + J^\sigma(u, v)$$

and

$$J(u, v) = (\nabla \cdot u, \nabla \cdot v).$$

2.1 Properties of the bilinear form

Now, we introduce a norm associated with the bilinear form B and set for $u \in H^1(\nabla\times, \Pi_h)$

$$\|u\|_h^2 = \|u\|^2 + \|c(\nabla \times u)\|^2 + \|\nabla \cdot u\|^2 + \left\| \frac{1}{\sqrt{\sigma}} \{\nabla \times u\} \right\|_{0,F_h}^2 + \|\sqrt{\sigma}[u]_N\|_{0,F_h^I}^2 + \|\sqrt{\sigma}[u]_T\|_{0,F_h}^2.$$

We start by studying the continuity of the bilinear forms introduced above. We have:

Proposition 2.1 $\forall v, u \in H^1(\nabla\times, \Pi_h)$ *there exists a constant* C *independent of* h *and* p *such that*

$$|A(u, v)| \le C\|u\|_h \|v\|_h \quad \textit{and} \quad |J^\sigma(u, v)| \le C\|u\|_h\|v\|_h.$$

Proof The proof is easily deduced from the definition of A, J^σ, $\|\cdot\|_h$ and the Cauchy-Schwarz inequality. □

In order to study the coercivity of the bilinear form B, we start by introducing the following inequality of Poincaré-Friedrichs type valid for $u \in H^1(\Pi_h)^3$.

Lemma 2.1 *Let* $u \in H^1(\Pi_h)^3$. *Then there exists* C *independent of* h *and* p *such that*

$$\|u\|^2 \le C\left(\|c(\nabla \times u)\|^2 + \|\nabla \cdot u\|^2 + \sum_{e\subset F_h} \|\sqrt{\sigma}[u]_T\|_{0,e}^2 + \sum_{e\subset F_h^I} \|\sqrt{\sigma}[u]_N\|_{0,e}^2 \right)$$

Proof The proof follows immediately from Lemma 3.1 in [4] since $\kappa p^2 \ge 1$. □

Now, the following coercivity result holds.

Proposition 2.2 *There exists two constants* $\alpha > 0$ *and* $\tilde{C} > 0$ *independent of* h *and* p *such that*

$$B(v, v) \ge \alpha\|v\|_h^2 + \tilde{C} J^\sigma(v, v), \quad \forall v \in \Sigma_h.$$

Proof Let us first recall the following inverse inequality

$$\|q\|_{0,\partial K}^2 \le C_{inv} \frac{p_K^2}{h_K} \|q\|_{0,K}^2, \quad \forall q \in S^{p_K}(K). \tag{2.1}$$

with a constant $C_{inv} > 0$, only depending on the shape regularity of the mesh. Now, Let α be an arbitrary real number and choose $v \in \Sigma_h$. Then

$$B(v, v) - \alpha\|v\|_h^2 = (1 - \alpha)(A(v, v) + J^\sigma(v, v)) - \alpha \int_{F_h} \{\nabla \times v\}^2/\sigma ds - \alpha\|v\|^2.$$

Since $\{\nabla \times v\}$ is the average of the flux at the face of two elements K_i and K_j, the corresponding integral can be split into two integrals with integrands $(\nabla \times v)_i/\sigma$ and $(\nabla \times v)_j/\sigma$, each one associated with the elements K_i or K_j respectively. Therefore, let $e \subset F_h$ and consider the integral associated with the element K. Using the inverse inequality (2.1), we have since $\nabla \times \ (\Sigma_h) \subset \Sigma_h$,

$$\int_e (\nabla \times v)^2/\sigma ds = \frac{1}{\sigma}\|\nabla \times v\|_{0,e}^2 \leq \frac{C_{inv}}{\sigma}\frac{p_K^2}{h_K}\|\nabla \times v\|_{0,K}^2 \tag{2.2}$$

so that, selecting σ to be equal to $\kappa p^2/h$ in (2.2), we obtain

$$-\int_e (\nabla \times v)^2/\sigma ds \geq -\frac{C_{inv}}{\kappa}\|\nabla \times v\|_{0,K}^2.$$

In particular,

$$\begin{aligned}-\int_{F_h} \{\nabla \times v\}^2/\sigma ds &\geq -\frac{C_{inv}}{\kappa}\sum_{K\in\Pi_h}\|\nabla \times v\|_{0,K}^2\\ &\geq -\frac{C_{inv}}{\kappa}A(v,v).\end{aligned}$$

It then follows that

$$B(v,v) - \alpha\|v\|_h^2 \geq \left(1 - \alpha - \alpha\frac{C_{inv}}{\kappa}\right)A(v,v) + (1-\alpha)J^\sigma(v,v) - \alpha\|v\|^2$$

and we can easily see there exists a positive constant C such that

$$\|u\|^2 \leq C(A(v,v) + J^\sigma(v,v)).$$

Then, we obtain

$$\begin{aligned}B(v,v) - \alpha\|v\|_h^2 &\geq (1 - \alpha C - \alpha - \alpha C_{inv}/\kappa)A(v,v) + (1 - \alpha - \alpha C)J^\sigma(v,v)\\ &\geq (1 - \alpha(C + 1 + C_{inv}/\kappa))A(v,v) + (1 - \alpha(1 + C))J^\sigma(v,v).\end{aligned}$$

Thus, if α is chosen for example $\alpha = \frac{1}{C+1+C_{inv}/\kappa}$ and $\tilde{C} = 1 - \frac{1+C}{C+1+C_{inv}/\kappa} > 0$, we immediately obtain the coercivity result. □

Now, the following *hp*-approximation result to interpolate scalar function holds (see [3]).

Proposition 2.3 *Let* $K \in \Pi_h$ *and suppose that* $u \in H^{t_K}(K)$, $t_K \geq 1$. *Then there exists a sequence of polynomials* $\pi_{p_K}^{h_K}(u) \in S^{p_K}(K)$, $p_K = 1, 2, \ldots$ *satisfying*, $\forall 0 \leq q \leq t_K$

$$\|u - \pi_{p_K}^{h_K}(u)\|_{q,K} \leq C\frac{h_K^{\min(p_K+1,t_K)-q}}{p_K^{t_K-q}}\|u\|_{t_K,K} \quad \text{and}$$

$$\|u-\pi_{p_K}^{h_K}(u)\|_{0,\partial K}\le C\frac{h_K^{\min(p_K+1,t_K)-\frac{1}{2}}}{p_K^{t_K-\frac{1}{2}}}\|u\|_{t_K,K}.$$

The constant C is independent of u, h_K and p_K, but depends on the shape regularity of the mesh.

In order to interpolate vector function, we define

Definition 2.1 For $u=(u_1,u_2,u_3)$ we define

$$\Pi_p^h: H^t(\nabla\times,\Pi_h)\longrightarrow\Sigma_h$$

by

$$\Pi_p^h(u)=(\pi_p^h(u_1),\pi_p^h(u_2),\pi_p^h(u_3))$$

with π_p^h is defined by $\pi_p^h(u)_{|K}=\pi_{p_K}^{h_K}(u_{|K})$ where $\pi_{p_K}^{h_K}$ is given by Proposition 2.3.

2.2 A priori error estimate

The interior penalty finite element approximation to u is to find $U: I\longrightarrow\Sigma_h$ such that

$$(U_{tt},v)+B(U,v)=(f,v),\quad\forall v\in\Sigma_h,\qquad U(0)=\Pi_p^h(u_0),\qquad U_t(0)=\Pi_p^h(u_1).\tag{2.3}$$

Upon choice of a basis for Σ_h and the data f, (2.3) determines U as the only solution to an initial value problem for a linear system of ordinary differential equations. Note that, if u is the exact solution of (1.1)–(1.2), then u satisfies the first equation in (2.3) and thus the problem is consistent.

We now analyse the proposed procedure by the method of energy estimates. In this section, u denotes the exact solution of (1.1)–(1.2) and U the discrete solution of (2.3). C is generic constant independent of h and p which takes different values at the different places and depends on α, $\tilde{C}$ the coercivity constants of the form B, t^* and Ω.

Let $\zeta=U-u$, then ζ satisfies

$$(\zeta_{tt},v)+B(\zeta,v)=0,\quad\forall v\in\Sigma_h.$$

Decompose ζ as $\mu-\nu$ where $\mu=\Pi_p^h(u)-u$ and $\nu=\Pi_p^h(u)-U$. Thus

$$(\nu_{tt},v)+B(\nu,v)=(\mu_{tt},v)+B(\mu,v),\quad\forall v\in\Sigma_h.$$

Since $\nu_t(t)\in\Sigma_h$, we can set $v=\nu_t(t)$, obtaining

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nu_t(t)\|^2+\frac{1}{2}\frac{d}{dt}B(\nu(t),\nu(t))&=(\mu_{tt}(t),\nu_t(t))+B(\mu(t),\nu_t(t))\\&\le\frac{1}{2}\|\mu_{tt}(t)\|^2+\frac{1}{2}\|\nu_t(t)\|^2+B(\mu(t),\nu_t(t)).\end{aligned}$$

So

$$\frac{d}{dt}\|\nu_t(t)\|^2 + \frac{d}{dt}B(\nu(t),\nu(t)) \le \|\mu_{tt}(t)\|^2 + \|\nu_t(t)\|^2 + 2B(\mu(t),\nu_t(t)).$$

Since $\nu_t(0) = \nu(0) = 0$, integration over $[0,t] \subset I$, yields

$$\|\nu_t(t)\|^2 + B(\nu(t),\nu(t)) \le \|\mu_{tt}\|^2_{L^2(L^2)} + \int_0^t \|\nu_t(t)\|^2 dt + 2\int_0^t B(\mu(t),\nu_t(t))dt.$$

The final term may be integrated by parts in time. Hence,

$$2\int_0^t B(\mu(t),\nu_t(t))dt \le 2|B(\mu(t),\nu(t))| + 2\int_0^t |B(\mu_t(t),\nu(t))|dt.$$

Therefore, we can apply the coercivity and continuity of B to get

$$\begin{aligned}
&\|\nu_t(t)\|^2 + \alpha\|\nu(t)\|_h^2 + \tilde{C}J^\sigma(\nu(t),\nu(t)) \\
&\quad \le \|\mu_{tt}\|^2_{L^2(L^2)} + \int_0^t \|\nu_t(t)\|^2 dt + C\|\nu(t)\|_h\|\mu(t)\|_h \\
&\qquad + 2\int_0^t |B(\mu_t(t),\nu(t))|dt \\
&\quad \le \|\mu_{tt}\|^2_{L^2(L^2)} + \int_0^t \|\nu_t(t)\|^2 dt + C\|\mu(t)\|_h^2 + \frac{\alpha}{2}\|\nu(t)\|_h^2 \\
&\qquad + C\int_0^t \left(\|\mu_t(t)\|_h^2 + \|\nu(t)\|_h^2\right) dt \\
&\quad \le C\left(\|\mu_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt\right) + \frac{\alpha}{2}\|\nu(t)\|_h^2 \\
&\qquad + C\int_0^t \left(\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2\right) dt.
\end{aligned}$$

In particular,

$$\begin{aligned}
&\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2 + J^\sigma(\nu(t),\nu(t)) \\
&\quad \le C\left(\|\mu_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt\right) \\
&\qquad + C\int_0^t \left(\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2\right) dt.
\end{aligned}$$

As this holds for all $t \in I$, Gronwall's Lemma implies that

$$\begin{aligned}
&\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2 + J^\sigma(\nu(t),\nu(t)) \\
&\quad \le C\left(\|\mu_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt\right).
\end{aligned}$$

Since $\zeta = \mu - \nu$ and $J^\sigma(\mu, \mu) = J^\sigma(\mu, \nu) = 0$,

$$\|\zeta_t(t)\|^2 + \|\zeta(t)\|_h^2 + J^\sigma(\zeta(t), \zeta(t)) \leq C\left(\|\mu_{tt}\|_{L^2(L^2)}^2 + \sup_{t\in I}\|\mu(t)\|_h^2 + \int_0^{t^*}\|\mu_t(t)\|_h^2 dt + \|\mu_t\|_{L^\infty(L^2)}^2\right).$$

Thus, error bounds for the finite element approximation to the true solution reduces to the error bounds for the piecewise polynomial interpolant. Thus, we start by estimating $\|u - \Pi_p^h(u)\|_h$ where Π_p^h is defined after Definition 2.1. By using Proposition 2.3 and the definition of $\|\cdot\|_h$, we obtain the following estimates

$$\|u - \Pi_p^h(u)\|_h^2 n \leq C \sum_{K\in\Pi_h} \frac{h_K^{2\mu_K-2}}{p_K^{2t_K-3}}\|u\|_{t_K,K}^2 \quad \text{and}$$

$$\|u - \pi_{p_K}^{h_K}(u)\|_{q,K} \leq C\frac{h_K^{\mu_K-q}}{p_K^{t_K-q}}\|u\|_{t_K,K}, \quad \forall 0 \leq q \leq t_K.$$

where $\mu_K = \min(p_K + 1, t_K)$. By using the previous estimates, we can get the following result

Proposition 2.4 *Let* $\mu_K = \min(p_K + 1, t_K)$ *and* u *be the exact solution of* (1.1)–(1.2). *Suppose that* $u_{|K} \in C^2(I, H^{t_K}(K)^3)$, $\forall K \in \Pi_h$ *with* $t_K \geq 2$. *Let* U *the discrete solution of* (2.3). *Then, the error* $\zeta = U - u$ *satisfies*

$$\|\zeta_t(t)\|^2 + \|\zeta(t)\|_h^2 + J^\sigma(\zeta(t), \zeta(t)) \leq C \sum_{K\in\Pi_h} \frac{h_K^{2\mu_K-2}}{p_K^{2t_K-3}}\big(\|u_{tt}\|_{L^2(H^{t_K}(K)^3)}^2 + \|u\|_{L^\infty(H^{t_K}(K)^3)}^2 + \|u_t\|_{L^2(H^{t_K}(K)^3)}^2 + \|u_t\|_{L^\infty(H^{t_K}(K)^3)}^2\big).$$

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